Example:- Show that weak Convergence does not imply Convergence in norm.

Soln:- By the application of the Riesz-representation theorem to the Hilbert Space $L_2(0, 2\pi)$, we find that

$$f(x) = \langle x, g \rangle = \int_0^{2\pi} x(t) g(t)\, dt \quad \text{--- (1)}$$

Consider the Sequence $\{x_n(t)\}$ defined below

$$x_n(t) = \frac{\sin nt}{\pi} \quad \text{for } n = 1, 2, 3, \text{----}$$

We Show now that $\{x_n(t)\}$ is weakly Convergent in $L_2(0, \pi)$ but is not norm convergent in $L_2(0, 2\pi)$.

Week Topologies week Convergence & Reflexive Spaces from eqn. (1), we have

$$f(x_n) = \langle x_n, g \rangle = \frac{1}{\pi} \int_0^{2\pi} \frac{\sin nt}{\pi} g(t)\, dt \quad \text{--- (2)}$$

The right hand side of eqn. (2) is the trigonometric fourier Coefficient of $g(t) \in L_2(0, \pi)$. By the Riemann-Lebesgue theorem Concerning the behaviour of trigonometric Fourier Coefficients

$$\frac{1}{\pi} \int_0^{2\pi} \sin nt\, g(t)\, dt \to 0 \text{ as } n \to \infty.$$

or $f(x_n) \to 0$ as $n \to \infty$, i.e. $f(x_n)$ Converges weakly to 0., we have,

$$\|x_n - 0\| = \|x_n\| = \left(\int_0^{2\pi} |x_n(t)|^2 dt\right)^{1/2}$$

$$= \left(\int_0^{2\pi} \frac{|\sin nt|^2}{\pi^2} dt\right)^{1/2} = \frac{1}{\sqrt{\pi}}$$

Since $\frac{1}{\pi}\left(\int_0^{2\pi} \sin^2 nt\, dt\right)^{1/2} \neq 0 \; \forall n$,

$\|x_n - 0\|$

Cannot tend to zero and therefore $\{x_n\}$ Cannot Converge in the norm.

Thus a weakly Convergent sequence need not be Convergent in the norm.

Example:- Show that weak* Convergence does not imply weak Convergence.

Soln:- Let $X = l_1$ (we know that l_1 is the dual of C_0 or m) and $Y = C_0$. Then $Y^* \cong l_1$.

Thus the dual of Y is X. Let $\{x^k\}$ be a sequence in X defined by the relation.

$$x_j^k = 0 \text{ if } k \neq j$$
$$= 1 \text{ if } k = j$$

For $y = (y_1, y_2, y_K \text{----}) \in C_0$, let $x^K(y) = y_K$

[x^K belongs to the dual of C_0, i.e. it is a bounded linear functional on C_0]

Since $y \in C_0$, $\underset{K \to \infty}{Lt.}\, y_K = 0$

and so $\underset{K \to \infty}{Lt.}\, x^K(y) = \underset{K \to \infty}{Lt.}\, y_K = 0$.

Therefore, the sequence $\{x^K\}$ of the dual space of $Y = C_0$ converges weak* to zero. Now if $Z \in X^* = l_\infty$ with $Z = (Z_1, Z_2 \text{---})$, then $x^K(Z) = Z_K$. Since $Z \in l_\infty$, $\{Z_K\}$ is bounded with respect to K but need not converge to zero as $K \to \infty$.

In fact, if $Z = (1, 1, \text{----})$ then $x^K(Z) \to 1$ as $K \to \infty$

Therefore, $\{x^K\}$ does not converge weakly.

This example shows that weak* convergence does not imply weak convergence.